# Equivariant splitting: self-supervised learning from incomplete data (ICLR'26)

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2026-03-05

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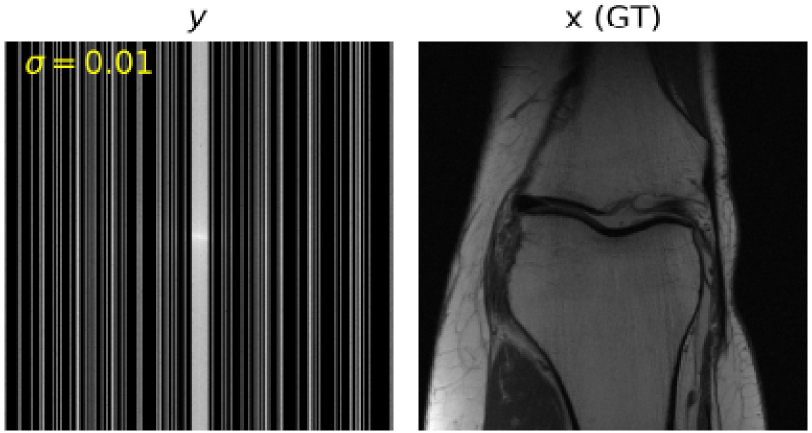
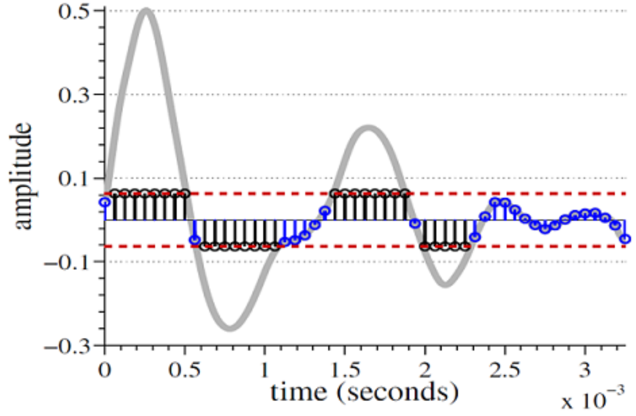



**Linear:** MRI, inpainting, ...



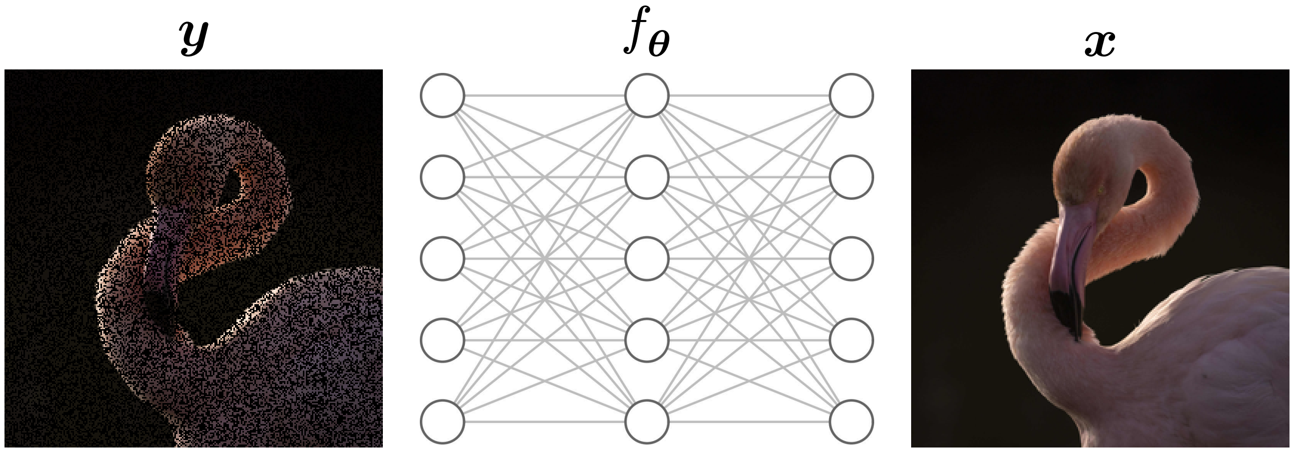

**Non-linear:** declipping, phase retrieval

# Deep learning approches

- Deep learning is now the state of the art.

## Objective

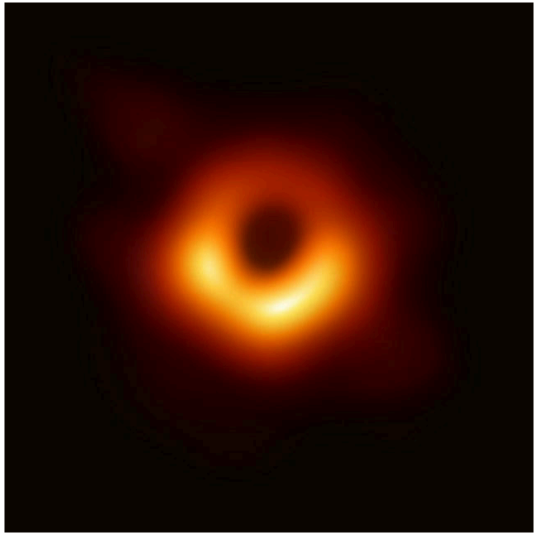
- **Learn** inverse forward operator of $\boldsymbol{A}$, $f_\theta(\boldsymbol{y}, \boldsymbol{A}) \approx \boldsymbol{x}$.
- **Using** a dataset $\{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i \in I}$ + a neural network.

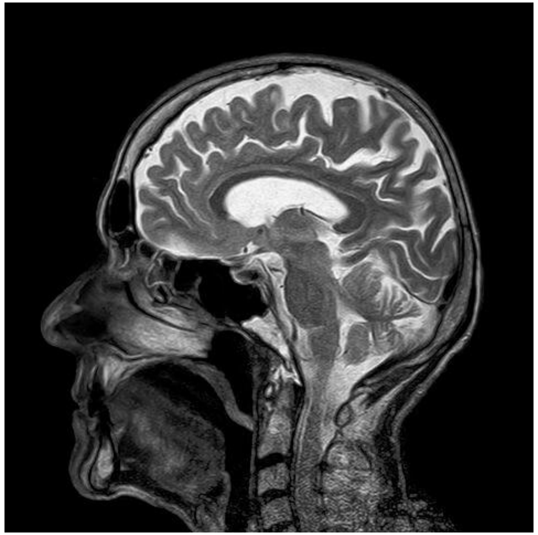



$$\underset{\theta}{\operatorname{argmin}} \sum_{i \in I} \|f_\theta(\boldsymbol{y}_i, \boldsymbol{A}) - \boldsymbol{x}_i\|^2$$

# Limits of the method

- Training and testing datasets can be very different.
- We need a large set of $\{\boldsymbol{x}_i\}_{i \in I}$ (ground-truth).





Astronomical and medical imaging.

# Measurement consistency is not enough

**Problem:** What can be done with a dataset of measurements only?

$$\mathcal{D} = (\boldsymbol{y}_i)_{i \in I} \neq (\boldsymbol{x}_i, \boldsymbol{y}_i)_{i \in I}$$

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$$\min_f \mathbb{E}_{\boldsymbol{y},\boldsymbol{A}}\left\{\|\boldsymbol{A}f(\boldsymbol{y},\boldsymbol{A}) - \boldsymbol{y}\|^2\right\}$$

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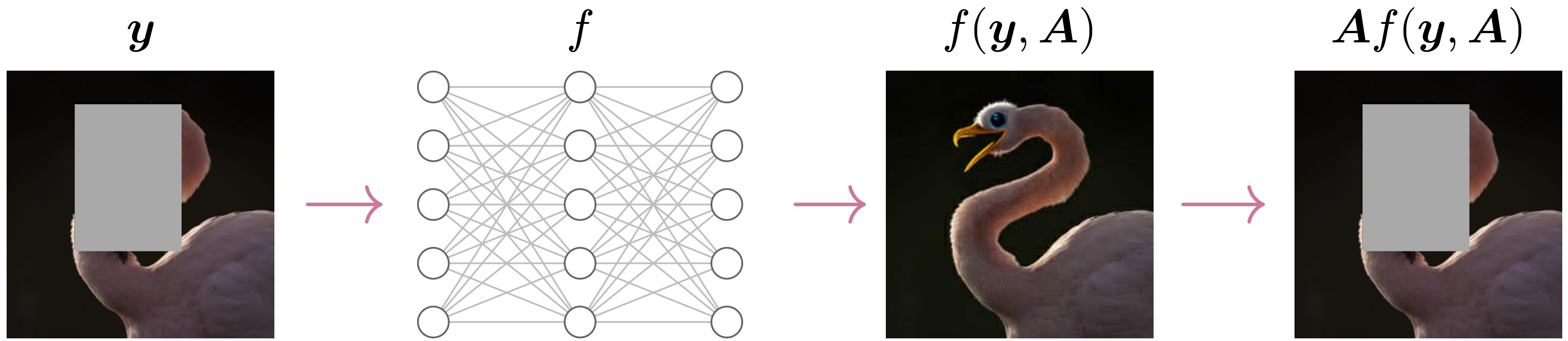
$$\min_f \mathbb{E}_{\boldsymbol{y}, \boldsymbol{A}} \left\{ \|\boldsymbol{A} f(\boldsymbol{y}, \boldsymbol{A}) - \boldsymbol{y}\|^2 \right\}$$

**Limitation:** It does not recover the information lost in the measurement process



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- Leveraging **multiple** operators: $\boldsymbol{A} \sim p(\boldsymbol{A})$.
- **Differ** for each measurements $\boldsymbol{y} \sim p(\boldsymbol{y} \mid \boldsymbol{A}\boldsymbol{x})$.

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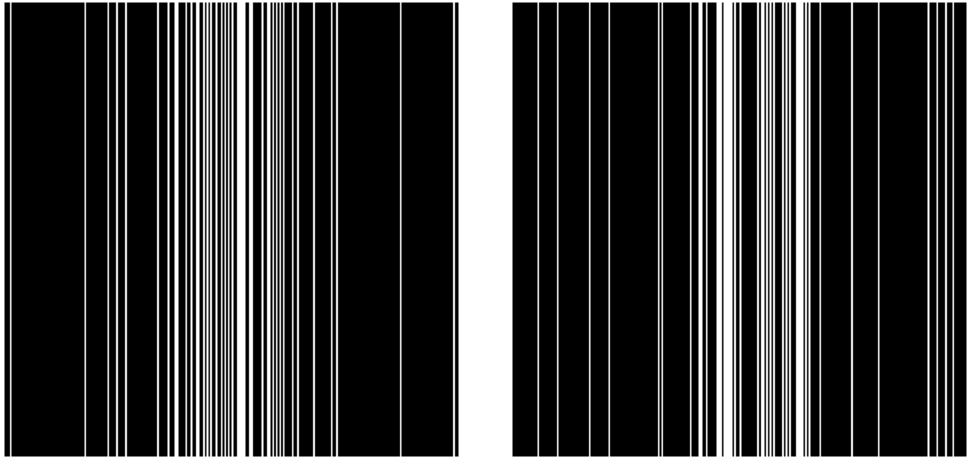

MRI

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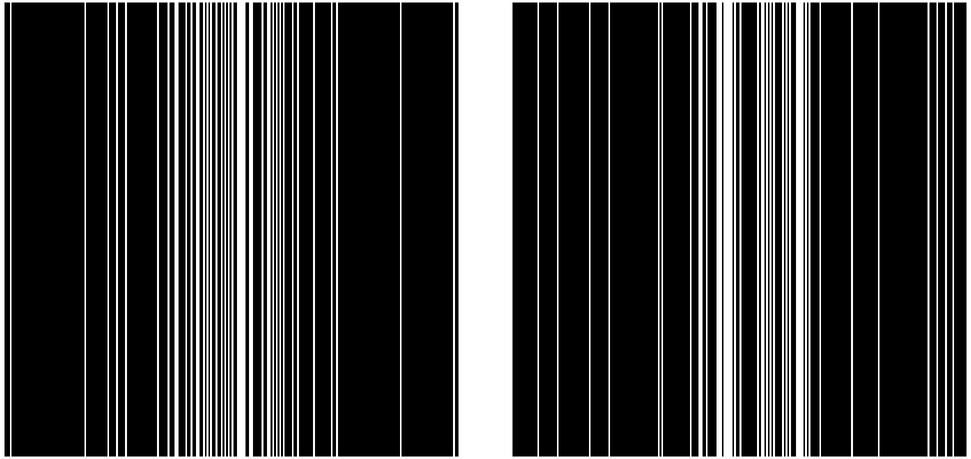

MRI

- **Measurement consistency** still learns the trivial reconstruction $f(\boldsymbol{y}, \boldsymbol{A}) = \boldsymbol{A}^{\dagger}\boldsymbol{y}$

# Measurements splitting

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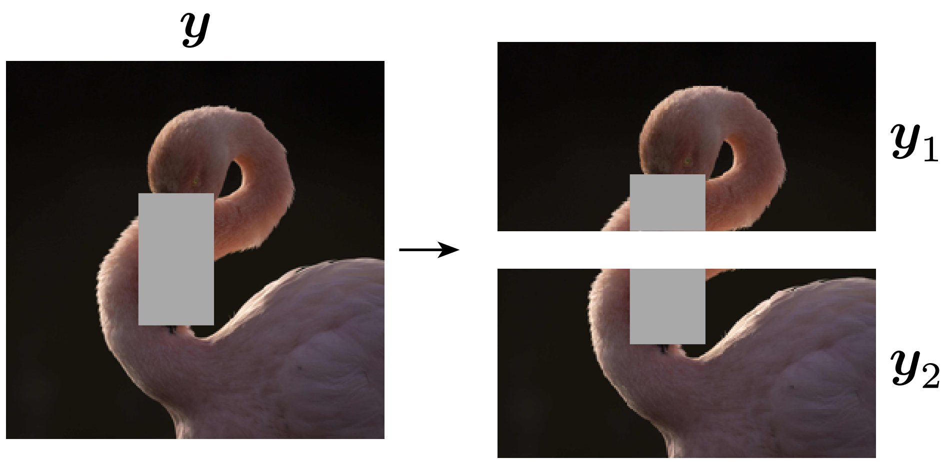
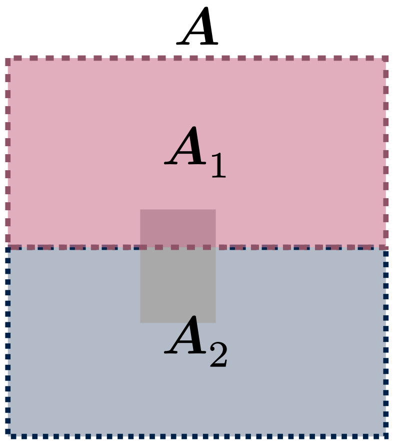
- **Split** measurements $\boldsymbol{y} = [\boldsymbol{y}_1, \boldsymbol{y}_2]$ with **corresponding** operators $\boldsymbol{A} = [\boldsymbol{A}_1, \boldsymbol{A}_2]$.

$$f^* \in \operatorname*{argmin}_f \mathbb{E}_{\boldsymbol{y},\boldsymbol{A}} \{\mathcal{L}_{\text{SPLIT}}(\boldsymbol{y}, \boldsymbol{A}, f)\}$$

$$\text{with} \quad \mathcal{L}_{\text{SPLIT}}(\boldsymbol{y}, \boldsymbol{A}, f) = \mathbb{E}_{\boldsymbol{y}_1, \boldsymbol{A}_1 \mid \boldsymbol{y}, \boldsymbol{A}} \left[ \left\| \boldsymbol{A} f(\boldsymbol{y}_1, \boldsymbol{A}_1) - \boldsymbol{y} \right\|^2 \right]$$



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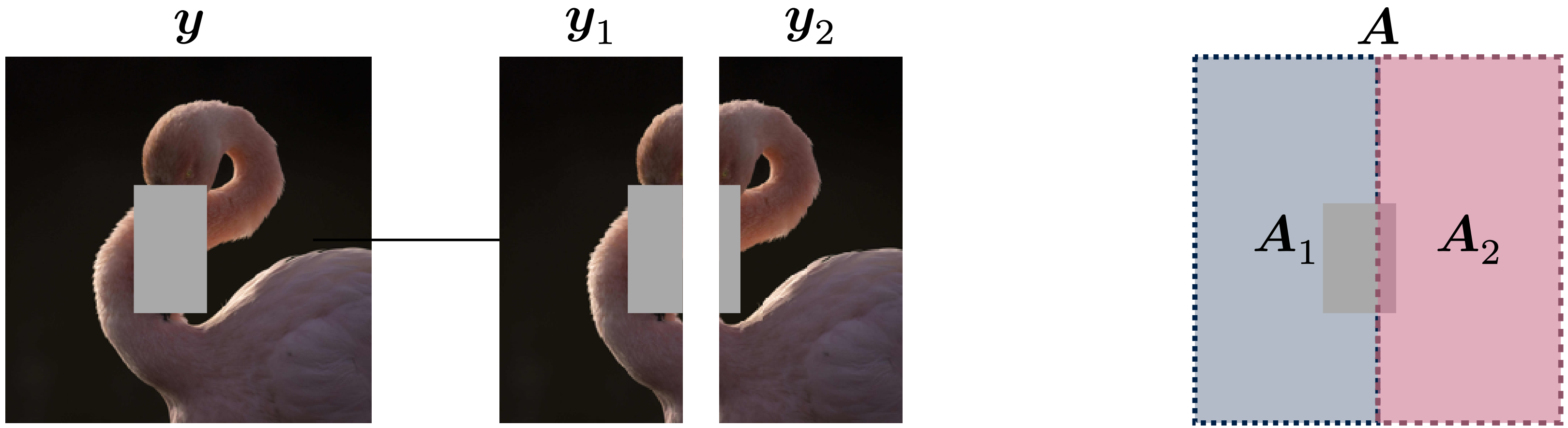

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**Solution** with $\boldsymbol{Q}_{\boldsymbol{A}_1} \triangleq \mathbb{E}_{\boldsymbol{A} \mid \boldsymbol{A}_1}\{\boldsymbol{A}^\top \boldsymbol{A}\}$ and $v$ any function:

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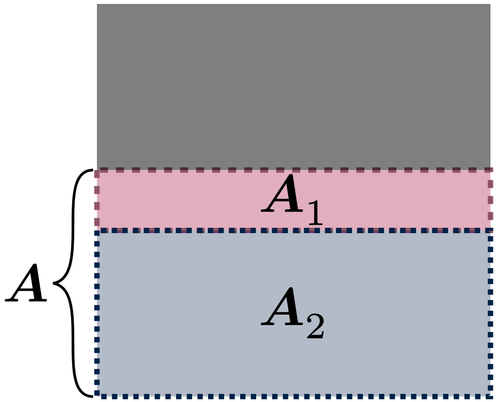

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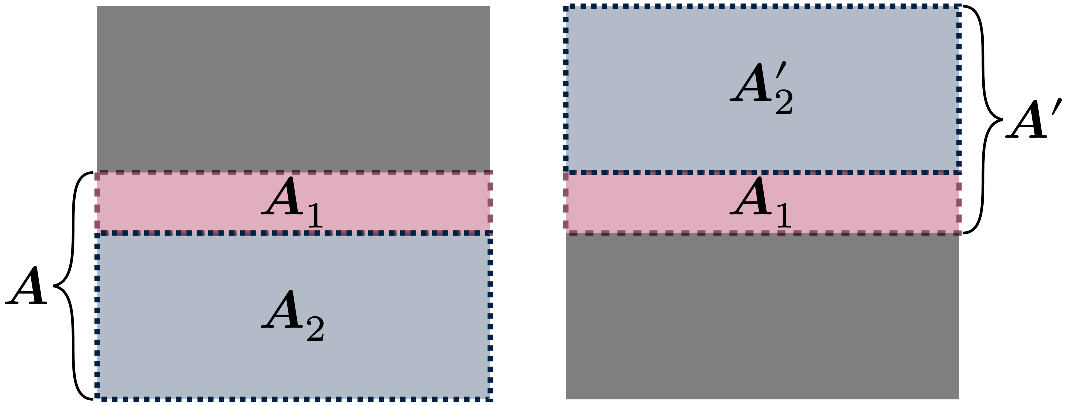

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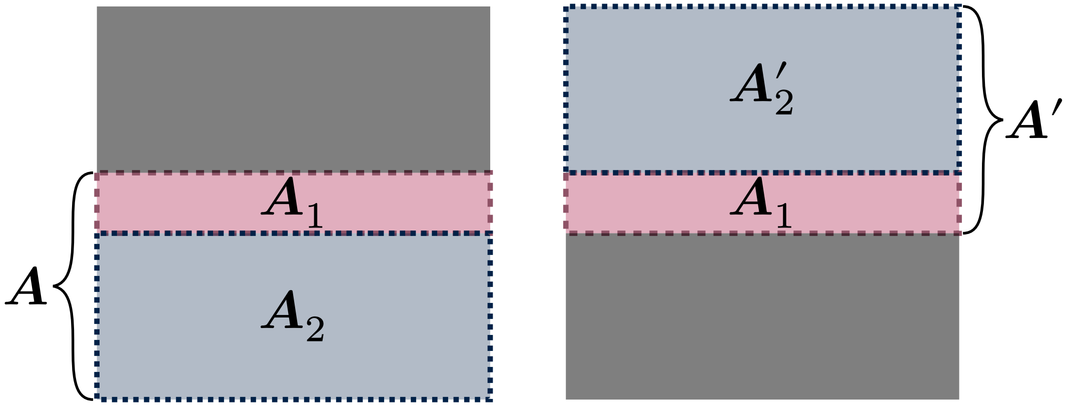


$\mathbb{E}_{\boldsymbol{A} \mid \boldsymbol{A}_1}[\boldsymbol{A}^\top \boldsymbol{A}]$

# Key ingredient: the invariances of image distributions

Can we learn if we have data from a **single operator** $A$?

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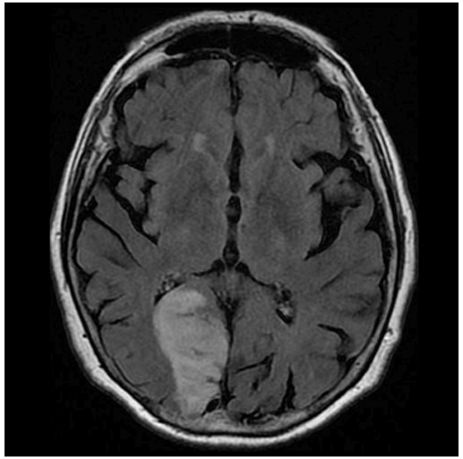
Can we learn if we have data from a **single operator** $\boldsymbol{A}$?

**Insight:** Information about the image distribution is needed

**Assumption:** The image distribution is invariant to certain transformations

$$p\left(\boldsymbol{T}_g\boldsymbol{x}\right) = p(\boldsymbol{x}), \quad \forall \boldsymbol{x}, g$$

$\boldsymbol{x}$



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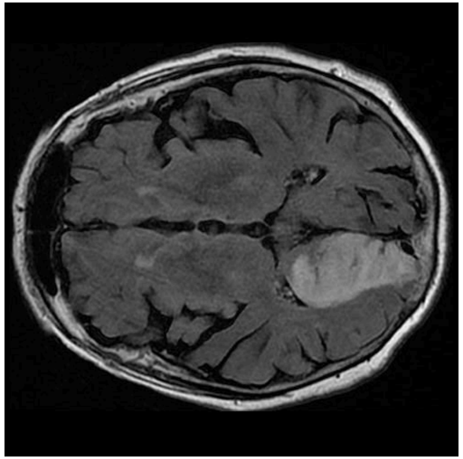

# Equivariant splitting

- **Virtual operators** from invariance: $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{T}_g \underbrace{\boldsymbol{T}_g^{-1}\boldsymbol{x}}_{\boldsymbol{x}' \in \mathcal{X}} = \boldsymbol{A}_g \boldsymbol{x}'$

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- **Split** measurements $\boldsymbol{y}$ for **different** operators $\boldsymbol{A}\boldsymbol{T}_g$:

$$\mathcal{L}_{\mathrm{ES}}(\boldsymbol{y}, \boldsymbol{A}, f) \triangleq \mathbb{E}_g \left\{ \mathcal{L}_{\mathrm{SPLIT}}\big(\boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g, f\big) \right\}$$

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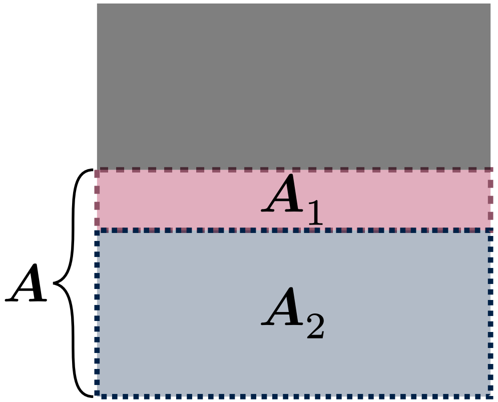

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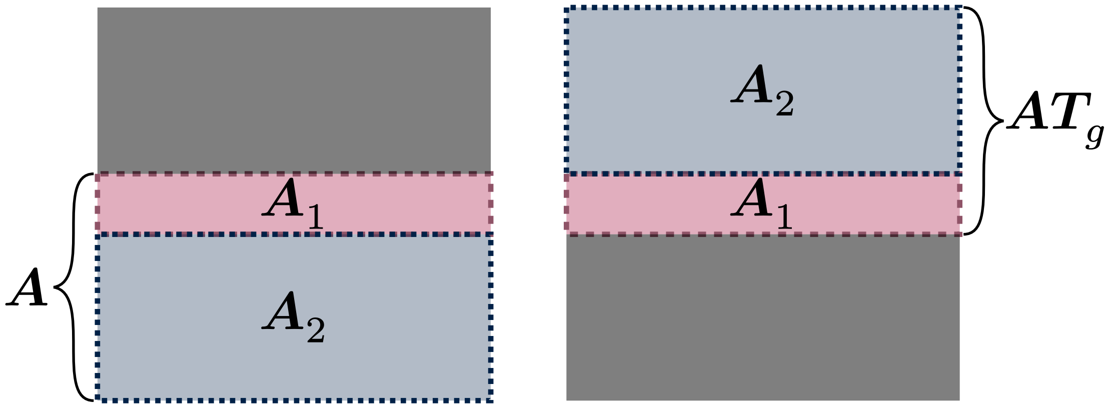

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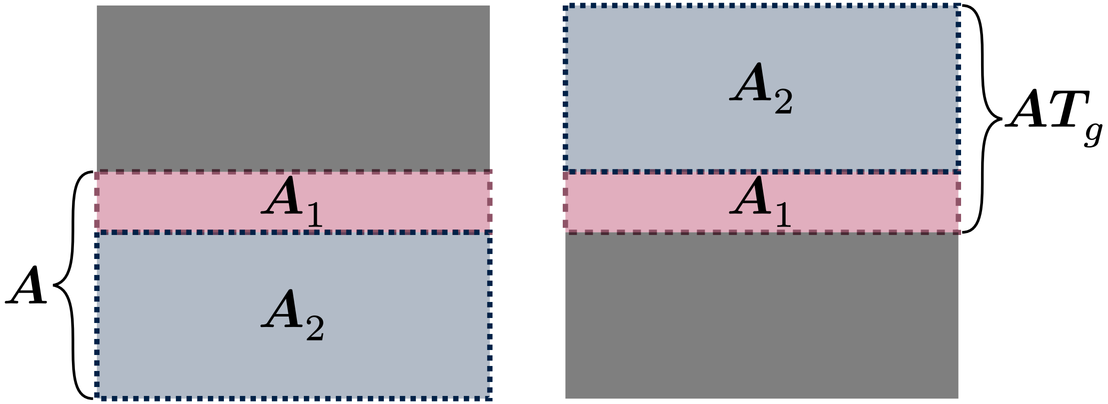


$$\mathbb{E}_{g \mid \boldsymbol{A}_1}\left[\left(\boldsymbol{A}\boldsymbol{T}_g\right)^\top \boldsymbol{A}\boldsymbol{T}_g\right]$$

# Equivariant reconstructors

**Definition:** A reconstructor $f(\boldsymbol{y}, \boldsymbol{A})$ is equivariant if it respects the equivalence

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon} \quad \Longleftrightarrow \quad \boldsymbol{y} = \left(\boldsymbol{A}\boldsymbol{T}_g\right)\left(\boldsymbol{T}_g^{-1}\boldsymbol{x}\right) + \boldsymbol{\varepsilon}$$

that is, if

$$f\left(\boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g\right) = \boldsymbol{T}_g^{-1} f(\boldsymbol{y}, \boldsymbol{A}), \quad \forall \boldsymbol{y}, \boldsymbol{A}, g$$

Moreover, if $p(\boldsymbol{x})$ is invariant, then **MMSE and MAP** are equivariant reconstructors.

# Design of equivariant reconstructors

**Artifact removal**

$$f(\boldsymbol{y}, \boldsymbol{A}) = \phi(\boldsymbol{A}^{+}\boldsymbol{y}) \quad \text{with } \phi(\cdot) \text{ satisfying} \quad \phi\big(\boldsymbol{T}_g x\big) = \boldsymbol{T}_g \phi(x)$$

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**Unrolled architecture**

$$x_{k+1} = \phi\Big(\boldsymbol{x}_k - \gamma \nabla_{\boldsymbol{x}_k}\Big\{\|\boldsymbol{A}\boldsymbol{x}_k - \boldsymbol{y}\|^2\Big\}\Big) \quad \text{with } \phi(\cdot) \text{ satisfying} \quad \phi\big(\boldsymbol{T}_g x\big) = \boldsymbol{T}_g \phi(x)$$

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**Reynolds averaging**

$$f(\boldsymbol{y}, \boldsymbol{A}) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \boldsymbol{T}_g \tilde{f}\big(\boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g\big) \quad \text{with } \tilde{f}(\boldsymbol{y}, \boldsymbol{A}) \text{ any reconstructor}$$

# Equivariance for faster compute

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**Problem.** Evaluating $\mathcal{L}_{\mathrm{ES}}(\boldsymbol{y}, \boldsymbol{A}, f)$ requires evaluating the network once per transform.

**Theorem.** If $f(\boldsymbol{y}, \boldsymbol{A})$ is an equivariant reconstructor, i.e.,

$$f\left(\boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g\right) = \boldsymbol{T}_g^{-1} f(\boldsymbol{y}, \boldsymbol{A}), \quad \forall \boldsymbol{y}, \boldsymbol{A}, g,$$

then

$$\mathcal{L}_{\mathrm{ES}}(\boldsymbol{y}, \boldsymbol{A}, f) = \mathcal{L}_{\mathrm{SPLIT}}(\boldsymbol{y}, \boldsymbol{A}, f).$$

# Inference

$$f^* \in \operatorname*{argmin}_f \mathbb{E}_{\boldsymbol{y}} \{ \mathcal{L}_{\text{ES}}(\boldsymbol{y}, \boldsymbol{A}, f) \}$$

- **At test time:** several splits $\boldsymbol{A}_1 \longrightarrow$ **average** the reconstructions.
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- If $\overline{\boldsymbol{Q}}_{\boldsymbol{A}} \triangleq \mathbb{E}_{\boldsymbol{A}_1 \mid \boldsymbol{A}}\left\{\boldsymbol{Q}_{\boldsymbol{A}_1}\right\}$ **is invertible**:

$$\begin{aligned}\overline{f}(\boldsymbol{y}, \boldsymbol{A}) &\triangleq \mathbb{E}_{\boldsymbol{y}_1, \boldsymbol{A}_1 \mid \boldsymbol{y}, \boldsymbol{A}}\left\{\overline{\boldsymbol{Q}}_{\boldsymbol{A}}^{-1} \boldsymbol{Q}_{\boldsymbol{A}_1} f^*(\boldsymbol{y}_1, \boldsymbol{A}_1)\right\} \\ &= \underbrace{\mathbb{E}_{\boldsymbol{y}_1, \boldsymbol{A}_1 \mid \boldsymbol{y}, \boldsymbol{A}}\left\{\overline{\boldsymbol{Q}}_{\boldsymbol{A}}^{-1} \boldsymbol{Q}_{\boldsymbol{A}_1} \mathbb{E}_{\boldsymbol{x} \mid \boldsymbol{y}_1, \boldsymbol{A}_1}\{\boldsymbol{x}\}\right\}}_{\text{Convex combination of MMSE estimators}}.\end{aligned}$$

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- If $\overline{\boldsymbol{Q}}_{\boldsymbol{A}}$ cannot be computed efficiently $\longrightarrow$ **non-weighted average**:

$$\overline{f}(\boldsymbol{y}, \boldsymbol{A}) = \frac{1}{J} \sum_{j=1}^{J} f^*\left(\boldsymbol{y}_1^{(j)}, \boldsymbol{A}_i^{(j)}\right) \quad \text{with} \quad \left(\boldsymbol{y}_1^{(j)}, \boldsymbol{A}_1^{(j)}\right) \sim p\big(\boldsymbol{y}_1, \boldsymbol{A}_1 \mid \boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g\big).$$

# Noisy measurements

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon} \quad \text{with} \quad \varepsilon \sim \mathcal{N}(0, \sigma^2 I)$$

- **Divide** the $\mathcal{L}_{\text{ES}}$ into 2 terms:

$$\mathcal{L}_{\text{ES}}(\boldsymbol{y}, \boldsymbol{A}, f) = \mathbb{E}_g \mathbb{E}_{\boldsymbol{y}_1, \boldsymbol{A}_1 \mid \boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g} \Big\{ \underbrace{\|\boldsymbol{A}_1 f(\boldsymbol{y}_1, \boldsymbol{A}_1) - \boldsymbol{y}_1\|^2}_{\text{Measurement consistency (MC)}} + \underbrace{\|\boldsymbol{A}_2 f(\boldsymbol{y}_1, \boldsymbol{A}_1) - \boldsymbol{y}_2\|^2}_{\text{prediction accuracy}} \Big\}.$$

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- **Replace** MC by a self-supervised **denoising** loss with $\mathcal{N}(0, \sigma^2 I)$ :

$$\mathbb{E}_g \mathbb{E}_{\boldsymbol{y}_1, \boldsymbol{A}_1, \boldsymbol{\omega} \mid \boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g} \Big\{ \underbrace{\Big\| \boldsymbol{A}_1 f(\boldsymbol{y}_1 + \alpha \boldsymbol{\omega}, \boldsymbol{A}_1) - \Big( \boldsymbol{y}_1 - \frac{\boldsymbol{\omega}}{\alpha} \Big) \Big\|^2}_{\text{R2R (denoising) loss}} + \|\boldsymbol{A}_2 f(\boldsymbol{y}_1 + \alpha \boldsymbol{\omega}, \boldsymbol{A}_1) - \boldsymbol{y}_2\|^2 \Big\}.$$

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- $\boldsymbol{Q}_{\boldsymbol{A}_1}$ invertible $\Rightarrow f^*(\boldsymbol{y}_1, \boldsymbol{A}_1) = \mathbb{E}_{\boldsymbol{x} \mid \boldsymbol{y}_1, \boldsymbol{A}_1}\{\boldsymbol{x}\}$.

# Overview of the experiments

| Modality | Network arch. (UNet) | Dataset |
|---|---|---|
| Compressive sensing | Shift-Equivariant (Unrolled) | MNIST |
| Image inpainting | Shift-Equivariant | DIV2K |
| MRI | Rotoflip-Equivariant (Unrolled) | FastMRI |
| CT | Rotoflip-Equivariant (Unrolled) | LIDC-IDRI |

- We compare with **Equivariant Imaging (EI)** (Chen, Davies & T., CVPR'22), which was the only self-supervised alternative in the single operator setting.

# ES is close to the supervised gold standard

**Reconstruction performance (PSNR in dB).**

| Method | Inpainting 30% rate | MRI ×8 accel. | CT 50 views |
|---|---|---|---|
| Supervised | 28.5 | 28.8 | 34.0 |
| ES | 27.5 | 28.3 | 32.6 |

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Ground Truth Supervised ES (ours) EI Measurement

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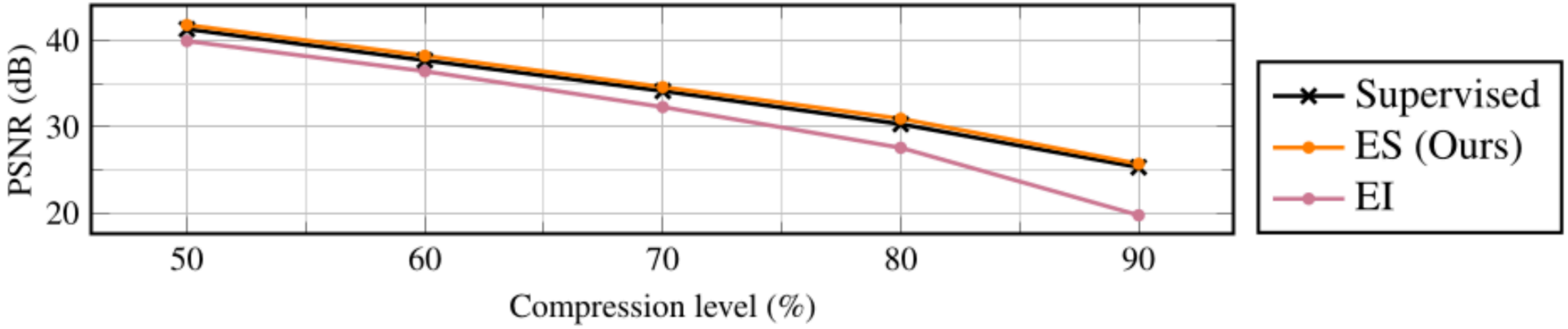
# ES vs EI for highly compressed measures




**Compressive sensing results.**

# Empirical benefit of equivariance

**Impact of equivariance (best in bold).**

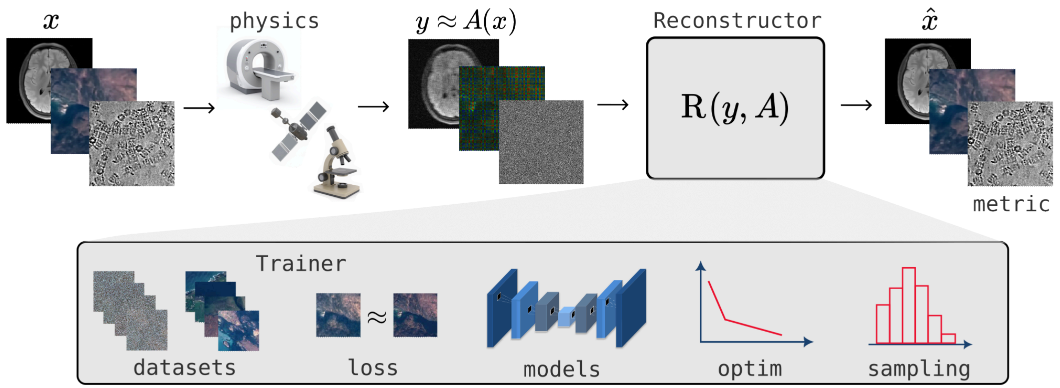
| | Inpainting 30% rate | | MRI ×8 accel. | |
|---|---|---|---|---|
| Equiv. arch. | PSNR ↑ | EQUIV ↑ | PSNR ↑ | EQUIV ↑ |
| ✅ | **27.5** | **27.5** | **28.5** | **31.5** |
| ❌ | 27.2 | 26.5 | 28.2 | 27.3 |

# Conclusion

- Self-supervised loss for learning from a **single incomplete** operator
- We show that **equivariant architectures** can enable self-supervised learning
- Theoretical guarantees on the learned estimator
- **Bonus:** Precise definition of equivariance in inverse problems

# DeepInverse

- Open-source library for computational imaging
- State-of-the-art reconstructors (PnP, diffusion, self-supervised, unrolling, etc)
- Annual hackathons and events




# Going deeper into self-supervised learning

- "Self-Supervised Learning from Noisy and Incomplete Data", Foundations & Trends in SP
- We provide a unified presentation of most self-sup methods
- Three hour tutorial available in YouTube https://youtu.be/gf-WCHXAdfk?/

# Code & acknowledgment

-**Code:** https://github.com/vsechaud/Equivariant-Splitting

-**MRI**: https://github.com/Andrewwango/ssibench

-**Photographs of flamingos:** Émile Sechaud