DEALing with Image Reconstruction: Deep Attentive Least Squares

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Inverse Problems in Imaging

Goal:
Estimate unknown image $\mathbf{x}$ given measurement $\mathbf{y}$.

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- Likelihood $p(\mathbf{y}|\mathbf{x})$ models forward measurement process

$$\mathbf{y} = H(\mathbf{x}) + \mathbf{n}$$

  $H\colon \mathcal{X} \to \mathcal{Y}$ forward operator
  $\mathbf{n}$ noise (e.g., Gaussian/Poisson noise, ...)

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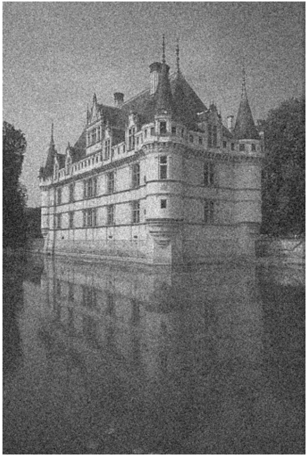
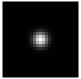
  $H\colon \mathcal{X} \to \mathcal{Y}$ forward operator
  $\mathbf{n}$ noise (e.g., Gaussian/Poisson noise, $\dots$)
- Prior $p(\mathbf{x})$ models a priori knowledge

Image Denoising



$$\xmapsto[\mathbf{n} \sim \mathcal{N}(0, \sigma^2 \mathrm{Id})]{\mathbf{H} = \mathrm{Id}}$$

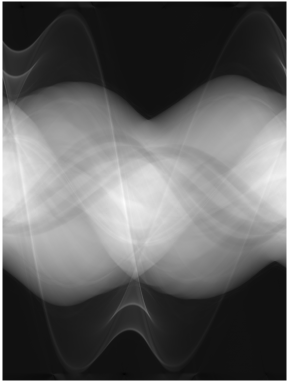
Image Superresolution



$$H\colon \mathbf{x} \mapsto (\mathbf{x} * \mathbf{a})_{\downarrow s}$$

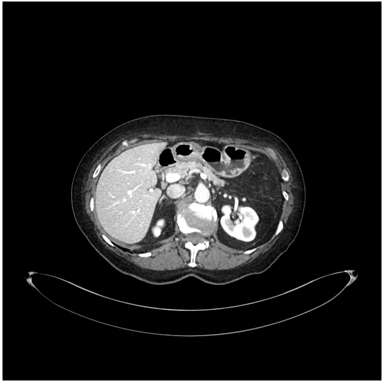
$$\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathrm{Id})$$

Computed Tomography (CT)

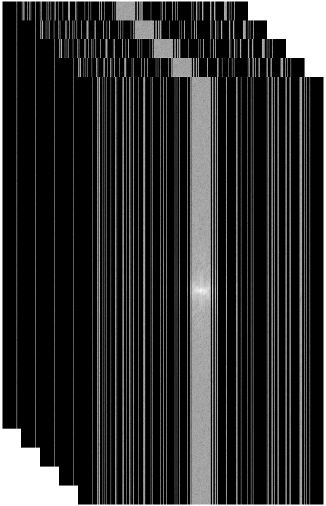


$$H_i : \mathbf{x} \mapsto \int_{\Gamma_i} \mathbf{x}(\xi)\mathrm{d}\xi$$

$$\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \Sigma)$$



Magnetic Resonance Imaging (MRI)



$$\xleftarrow[\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathrm{Id})]{H_i : \mathbf{x} \mapsto \mathbf{m} \odot \mathcal{F}(\mathbf{c} \odot \mathbf{x})}$$

Variational approach:

Find image maximizing posterior

$$\hat{\mathbf{x}} = \operatorname*{argmax}_{\mathbf{x}} p(\mathbf{x}|\mathbf{y})$$

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Variational Methods for Inverse Problems

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Data fidelity:
For linear inverse problem and homoscedastic Gaussian noise, simplifies to

$$D(\mathbf{x}, \mathbf{y}) = - \log p(\mathbf{y}|\mathbf{x}) = \frac{1}{2\sigma^2} \|\mathbf{H}\mathbf{x} - \mathbf{y}\|_2^2$$

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Regularizer:
Impose prior knowledge on solution $\mathbf{x}$

$$R(\mathbf{x}) = -\log p(\mathbf{x})$$

Regularizers

- Tikhonov regularization[1]

$$R(\mathbf{x}) = \frac{1}{2}\|\mathbf{Wx}\|_2^2 = \left\langle \mathbf{1}, (\mathbf{Wx})^2 \right\rangle$$

[1] Tikhonov, Arsenin (1977) *Solutions of Ill-Posed Problems*.

[2] Rudin, Osher, Fatemi (1992). *Nonlinear total variation based noise removal algorithms*. Physica D.

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- Total variation[2] (TV) regularization

$$R(\mathbf{x}) = \left\|\mathbf{W}\mathbf{x}\right\|_{2,1} = \left\langle \mathbf{1}, \sqrt{(\mathbf{W}_h\mathbf{x})^2 + (\mathbf{W}_v\mathbf{x})^2} \right\rangle$$

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- Fields of experts[3] (FoE)

$$R(\mathbf{x}) = \sum_{c=1}^{N_c} \langle \mathbf{1}, \psi_c(\mathbf{W}_c\mathbf{x}) \rangle$$

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Computing Solutions

- Tikhonov regularization $R(\mathbf{x}) = \frac{1}{2}\|\mathbf{Wx}\|_2^2$

$$\hat{\mathbf{x}} = \operatorname*{argmin}_{\mathbf{x}} \left\{ \frac{1}{2\sigma^2}\|\mathbf{Hx}-\mathbf{y}\|_2^2 + \frac{1}{2}\|\mathbf{Wx}\|_2^2 \right\}$$

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Iterative solvers such as proximal (sub)gradient descent

$$\mathbf{x}_{k+1} = \mathrm{prox}_{\tau_k D}(\mathbf{x}_k - \tau_k \sum_{c=1}^{N_c} \mathbf{W}_c^\top \nabla\psi_c(\mathbf{W}_c\mathbf{x}))$$

using $\mathrm{prox}_{\tau D}(\mathbf{y}) = \mathrm{argmin}_{\mathbf{x}} \left\{ \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_2^2 + \frac{\tau\lambda}{2}\|\mathbf{H}\mathbf{x}-\mathbf{y}\|_2^2 \right\}$

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Deep (implicit) Regularization

- Total Deep Variation[4] regularization

$$R(\mathbf{x}) = \langle \mathbf{1}, \psi(\mathbf{W}_{\theta}(\mathbf{x})) \rangle$$

[4] Kobler et al. (2020) *Total Deep Variation for Linear Inverse Problems*. CVPR.

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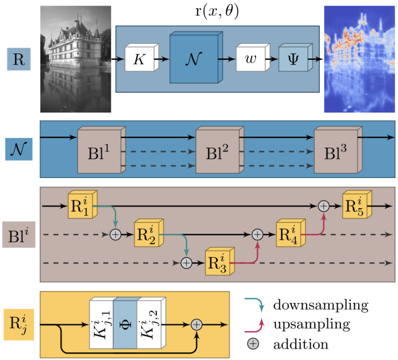

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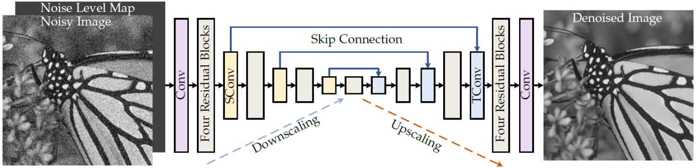
- Total Deep Variation[5] regularization

$$R(\mathbf{x}) = \langle \mathbf{1}, \phi(\mathbf{W}_\theta(\mathbf{x})) \rangle$$

- DRUNet[6] plug & play regularization

$$\operatorname{prox}_R^\sigma(\mathbf{y}) = \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{y}\|_2^2 + R(\mathbf{x}) \right\} \approx \mathbf{N}_\theta(\mathbf{y}, \sigma)$$



[5] Kobler et al. (2020) *Total Deep Variation for Linear Inverse Problems*. CVPR.
[6] Zhang et al. (2021) *Plug-and-Play Image Restoration with Deep Denoiser Prior*. TPAMI.

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- Prox-DRUNet[9] gradient step denoiser as proximal operator

$$\mathbf{D}_\sigma = \mathrm{Id} - \nabla g_\theta(\cdot, \sigma) \overset{\nabla g \text{ contractive}}{=} \operatorname{prox}_{\phi_\sigma}$$

[7] Kobler et al. (2020) *Total Deep Variation for Linear Inverse Problems*. CVPR.

[8] Zhang et al. (2021) *Plug-and-Play Image Restoration with Deep Denoiser Prior*. TPAMI.

[9] Hurault et al. (2022) *Proximal Denoiser for Convergent Plug-and-Play Optimization with Nonconvex Regularization*. ICML.

FoE regularization

$$\sum_{c=1}^{N_c} \langle \mathbf{1}, \psi_c(\mathbf{W}_c \mathbf{x}) \rangle$$

[9] Pourya, Kobler, Unser, Neumayer (2025) *DEALing with Image Reconstruction: Deep Attentive Least Squares*. ICML.

Deep Attentive Least Squares (DEAL)

FoE regularization

$$\sum_{c=1}^{N_c} \langle \mathbf{1}, \psi_c(\mathbf{W}_c\mathbf{x}) \rangle$$

$$\Downarrow$$

$$\sum_{c=1}^{N_c} \left\langle \mathbf{m}_c{}^2, (\mathbf{W}_c\mathbf{x})^2 \right\rangle$$

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$$\sum_{c=1}^{N_c} \left\langle {\mathbf{m}_c}^2, (\mathbf{W}_c\mathbf{x})^2 \right\rangle \quad = \quad \|\mathbf{W}\mathbf{x}\|^2_{\mathbf{m}^2} =: R(\mathbf{x})$$

DEAL regularization

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Deep Attentive Least Squares (DEAL)

Role of local weights m

Adapt regularization strength to local image structure.

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How to compute local weights m?

- Ideally, reflect structure of unkown ground truth.
- Idea: use current best estimate $\mathbf{x}_k$ to compute $\mathbf{m}_k = \mathbf{M}(\mathbf{x}_k)$ using CNN

$$\mathbf{M}(\mathbf{x}) = (\phi^{\sigma} \circ \mathbf{W}^2_{\mathrm{mix}} \circ \varphi_2 \circ \mathbf{W}^1_{\mathrm{mix}} \circ \varphi_1 \circ \mathbf{W}_{\mathrm{mask}})(\mathbf{x})$$

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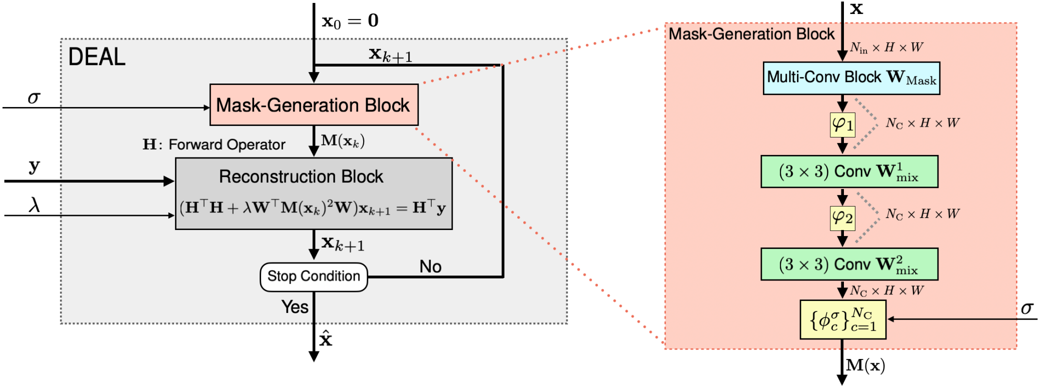
Fixedpoint formulation $\mathbf{x}_{k+1} \in \mathcal{T}(\mathbf{x}_k, \mathbf{y}) \quad \text{with} \quad \mathbf{x}_0 = \mathbf{0}$

$$\mathcal{T}(\mathbf{x}_k, \mathbf{y}) = \operatorname*{argmin}_{\mathbf{x}} \left(\|\mathbf{Hx} - \mathbf{y}\|_2^2 + \lambda \, \|\mathbf{Wx}\|^2_{\mathbf{M}(\mathbf{x}_k)} \right)$$

$$= \left(\mathbf{H}^\top \mathbf{H} + \lambda \mathbf{W}^\top \mathbf{M}(\mathbf{x}_k)^2 \mathbf{W} \right)^{-1} \mathbf{H}^\top \mathbf{y} \iff \mathbf{A}_k \mathbf{x} = \mathbf{b}$$

DEAL Algorithm



DEAL – Training

On image denoising ($\mathbf{H} = \mathrm{Id}$)

$$\mathcal{L}(\theta) = \left\{ \mathbb{E}_{\substack{\mathbf{x}\sim \mathrm{D} \\ \sigma_{\mathrm{n}}\sim\mathcal{U}([0,50]) \\ \mathbf{n}\sim\mathcal{N}(\mathbf{0},\mathrm{Id})}} \left[\|\mathbf{x}_{K_{\mathrm{out}}} - \mathbf{x}\|_2^2 + \frac{\gamma}{N_{\mathrm{C}}} \|\mathbf{M}(\mathbf{x}_{K_{\mathrm{out}}}) - \mathbf{M}(\mathbf{x}_{K_{\mathrm{out}}-1})\|_2^2 \right] + \gamma \mathrm{TV}^2(\theta) \right\}$$

using $K_{\mathrm{out}} \in \{15, \ldots, 60\}$

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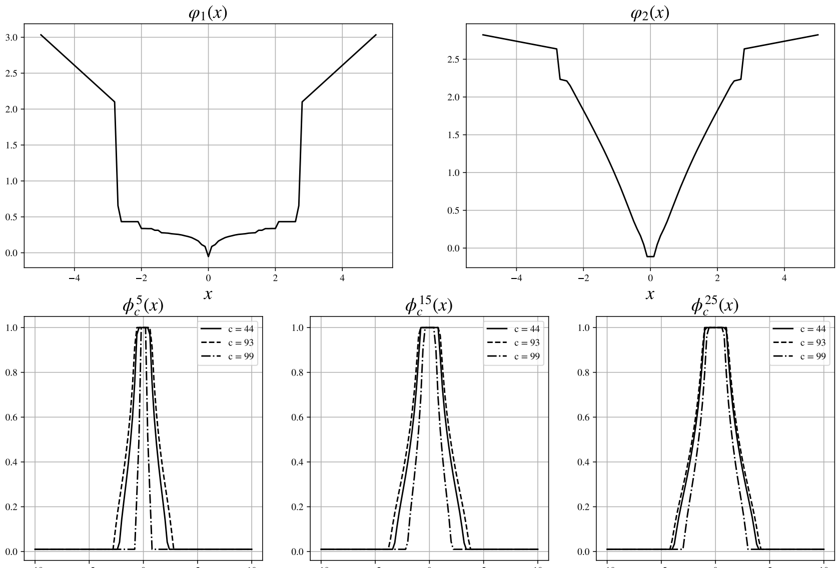
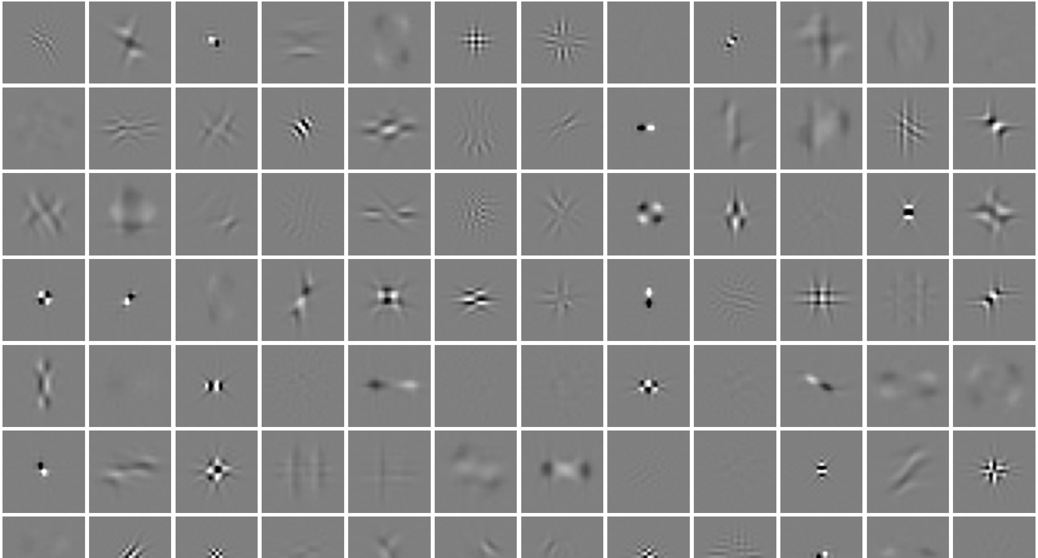
Backpropagation though CG by implicit differentiation

$$\mathbf{A}_k \partial_\theta \mathbf{x}_{k+1} = -\lambda \partial_\theta \left(\mathbf{W}^\top \mathbf{M}(\mathbf{x}_k) \mathbf{W} \right) \mathbf{x}_{k+1}$$

Learned Functions

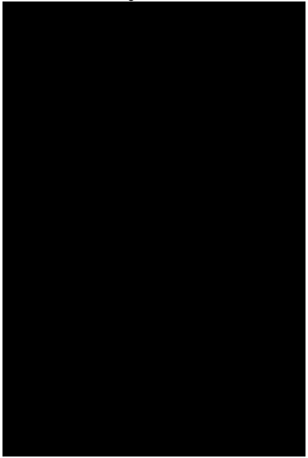


Learned Kernels

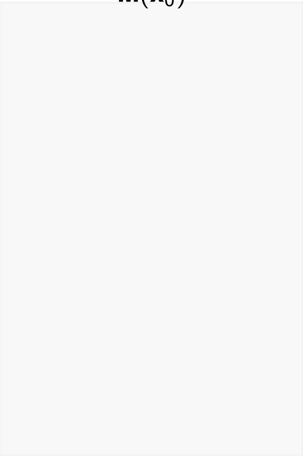


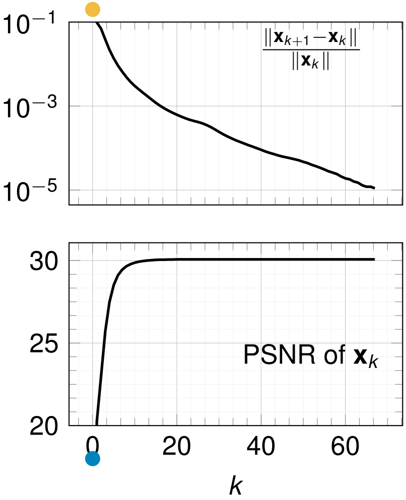
Results Image Denoising

$\mathbf{x}_0$ 6.12



$\overline{\mathbf{M}}(\mathbf{x}_0)$

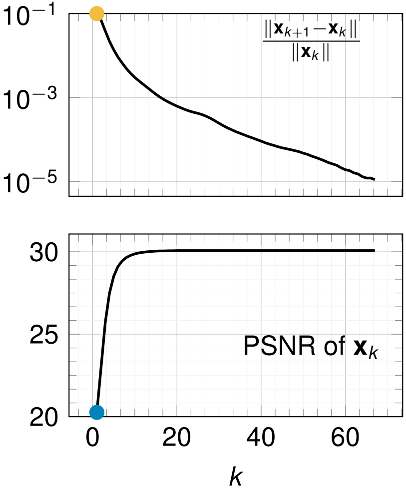




Results Image Denoising

$\mathbf{x}_1$ 20.26

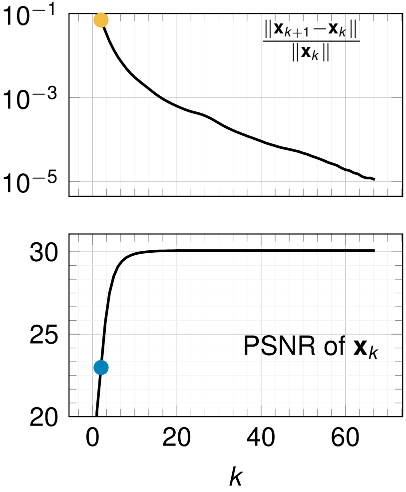
$\overline{\mathbf{M}}(\mathbf{x}_1)$



Results Image Denoising

$\mathbf{x}_2$ 22.98

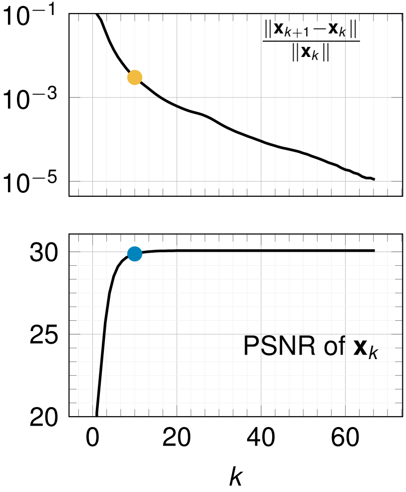
$\overline{\mathbf{M}}(\mathbf{x}_2)$



Results Image Denoising

$\mathbf{x}_{10}$ 29.87

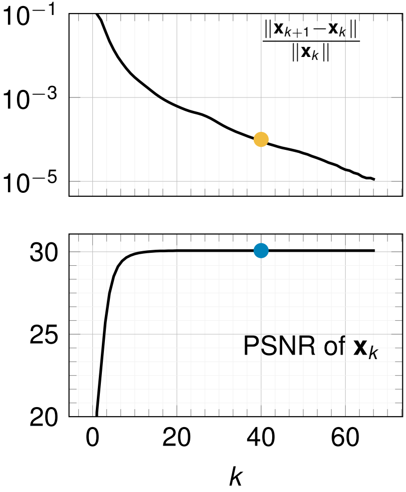
$\overline{\mathbf{M}}(\mathbf{x}_{10})$



Results Image Denoising

$\mathbf{x}_{40}$ 30.07

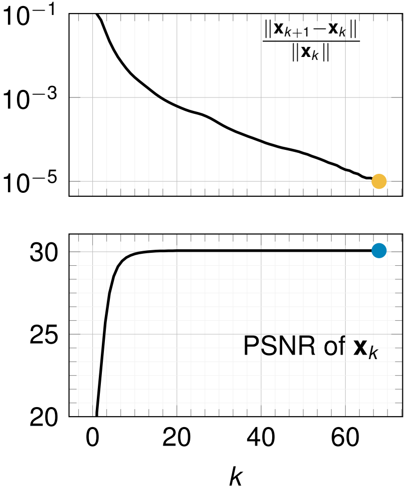
$\overline{\mathbf{M}}(\mathbf{x}_{40})$



Results Image Denoising

$\mathbf{x}_{68}$ 30.07

$\overline{\mathbf{M}}(\mathbf{x}_{68})$



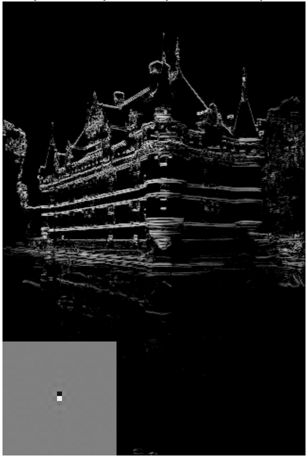
Denoising of BSD68 and CBSD68 images

	Gray			Color		
σ_{n}	5	15	25	5	15	25
BM3D	37.54	31.13	28.61	40.19	33.52	30.71
WCRR	37.65	31.20	28.68	–	–	–
SAFI	37.90	31.56	29.05	–	–	–
DEAL (Ours)	37.85	31.61	29.16	40.33	<u>33.95</u>	<u>31.31</u>
ProxDRUNet	<u>37.97</u>	31.70	29.18	<u>40.40</u>	33.91	31.14
DnCNN	–	<u>31.72</u>	<u>29.23</u>	-	33.90	31.24
DRUNet	**38.09**	**31.94**	**29.48**	**40.59**	**34.30**	**31.69**

DEAL – Effect of Local Weights

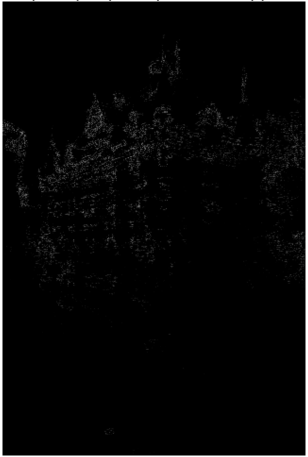
Filter channel $c = 93$

$(\mathbf{W}_c \mathbf{x}_k)^2 = (\mathbf{W}_c * \mathbf{x}_k)^2$



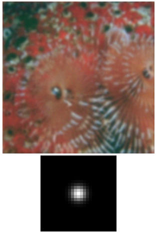
$\mathbf{M}_c(\mathbf{x}_k)$

$(\mathbf{M}_c(\mathbf{x}_k) \odot (\mathbf{W}_c * \mathbf{x}_k))^2$



DEAL - Single-Image Superresolution

$$\mathcal{T}(\mathbf{x}_k, \mathbf{y}) = \operatorname*{argmin}_{\mathbf{x}} \left(\|\mathbf{H}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{W}\mathbf{x}\|^2_{\mathbf{M}(\mathbf{x}_k)} \right)$$



$$H : \mathbf{x} \mapsto (\mathbf{x} * \mathbf{a})_{\downarrow s}$$

$$\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathrm{Id})$$

Results Single-Image Superresolution

SISR results on CBSD68 dataset

	Category	$\#\theta$	PSNR (s = 2)				PSNR (s=3)			
			<u>bicubic</u>	<u>4 different kernels</u>			<u>bicubic</u>	<u>4 different kernels</u>		
Noise σ_{n}			0	2.55	7.65	12.75	0	2.55	7.65	12.75
DEAL	Explicit Reg	<u>0.85</u>	<u>29.91</u>	**27.99**	<u>26.58</u>	25.75	<u>26.83</u>	**26.20**	<u>25.27</u>	24.59
Prox-PnP	Conv. PnP	32.64	-	<u>27.93</u>	**26.61**	<u>25.79</u>	-	<u>26.13</u>	**25.29**	**24.67**
IRCNN	PnP	**0.19**	29.84	26.97	25.86	25.45	26.74	25.60	25.72	24.38
DPIR	PnP	32.64	29.63	27.79	<u>26.58</u>	**25.83**	26.70	26.05	<u>25.27</u>	<u>24.66</u>
DiffPIR [4]	Diff. Model	93.56	29.73	27.84	26.48	25.63	-	-	-	-
SwinIR [3]	End to End	11.75	**30.88**	24.56	22.84	20.73	**27.76**	22.41	21.24	19.53

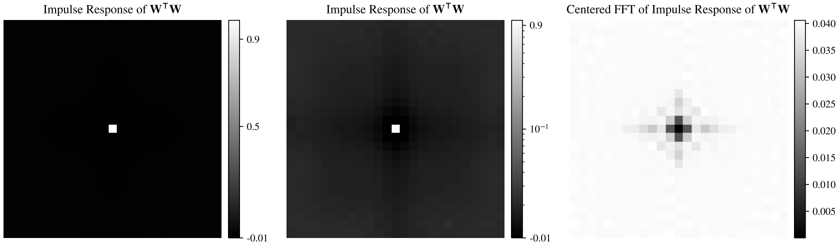
Theorem: Fixedpoint Existence
Assume that $\ker(\mathbf{H}) \cap \ker(\mathbf{W}) = \{\mathbf{0}\}$ and $\mathbf{M}^2(\mathbf{x}) \succeq \epsilon_M \mathrm{Id}$. Then, $\mathcal{T}(\cdot, \mathbf{y})\colon \mathbb{R}^d \to B_r(\mathbf{0})$ maps into a ball around $\mathbf{0}$ with radius $r = \|\mathbf{Hy}\|_2 / \lambda_\epsilon$, where $\lambda_\epsilon = \lambda_{\min}(\mathbf{A}_k)$. If $\mathbf{M}^2\colon B_r(\mathbf{0}) \to [\epsilon, 1]^{N_{\mathrm{out}}HW}$ is Lipschitz continuous with constant L, then $\mathcal{T}(\cdot, \mathbf{y})$ admits a fixed point and

$$\|\mathcal{T}(\mathbf{x}_1, \mathbf{y}) - \mathcal{T}(\mathbf{x}_2, \mathbf{y})\|_2 \leq \frac{L\,\|\mathbf{Hy}\|_2}{\lambda_\epsilon^2} \|\mathbf{x}_1 - \mathbf{x}_2\|_2.$$

DEAL – Analysis

Theorem: Fixedpoint Existence
Assume that $\ker(\mathbf{H}) \cap \ker(\mathbf{W}) = \{\mathbf{0}\}$ and $\mathbf{M}^2(\mathbf{x}) \succeq \epsilon_M \mathrm{Id}$. Then, $\mathcal{T}(\cdot, \mathbf{y})\colon \mathbb{R}^d \to B_r(\mathbf{0})$ maps into a ball around $\mathbf{0}$ with radius $r = \|\mathbf{Hy}\|_2 / \lambda_\epsilon$, where $\lambda_\epsilon = \lambda_{\min}(\mathbf{A}_k)$. If $\mathbf{M}^2\colon B_r(\mathbf{0}) \to [\epsilon, 1]^{N_{\text{out}}HW}$ is Lipschitz continuous with constant L, then $\mathcal{T}(\cdot, \mathbf{y})$ admits a fixed point and

$$\|\mathcal{T}(\mathbf{x}_1, \mathbf{y}) - \mathcal{T}(\mathbf{x}_2, \mathbf{y})\|_2 \leq \frac{L\,\|\mathbf{Hy}\|_2}{\lambda_\epsilon^2} \|\mathbf{x}_1 - \mathbf{x}_2\|_2.$$



Theorem: Unique Fixedpoint
Assume that $\ker(\mathbf{H}) \cap \ker(\mathbf{W}) = \{\mathbf{0}\}$ and $\mathbf{M}^2(\mathbf{x}) \succeq \epsilon_M \mathrm{Id}$. If $\mathcal{T}(\cdot, \mathbf{y})\colon \mathbb{R}^d \to \mathbb{R}^d$ is contractive in the sense that if $\|\mathcal{T}(\mathbf{x}_1, \mathbf{y}) - \mathcal{T}(\mathbf{x}_2, \mathbf{y})\|_2 \leq q\|\mathbf{x}_1 - \mathbf{x}_2\|_2$ with $q < 1$ implies that the iterations converge to a unique fixed point $\hat{\mathbf{x}}$ and that

$$\|\mathbf{x}_k - \hat{\mathbf{x}}\|_2 \leq q^{k-1}\|\mathbf{x}_1 - \mathbf{x}_0\|_2.$$

In particular, we have exponential convergence.

Theorem: Unique Fixedpoint
Assume that $\ker(\mathbf{H}) \cap \ker(\mathbf{W}) = \{\mathbf{0}\}$ and $\mathbf{M}^2(\mathbf{x}) \succeq \epsilon_M \mathrm{Id}$. If $\mathcal{T}(\cdot, \mathbf{y}) \colon \mathbb{R}^d \to \mathbb{R}^d$ is contractive in the sense that if $\|\mathcal{T}(\mathbf{x}_1, \mathbf{y}) - \mathcal{T}(\mathbf{x}_2, \mathbf{y})\|_2 \leq q\|\mathbf{x}_1 - \mathbf{x}_2\|_2$ with $q < 1$ implies that the iterations converge to a unique fixed point $\hat{\mathbf{x}}$ and that

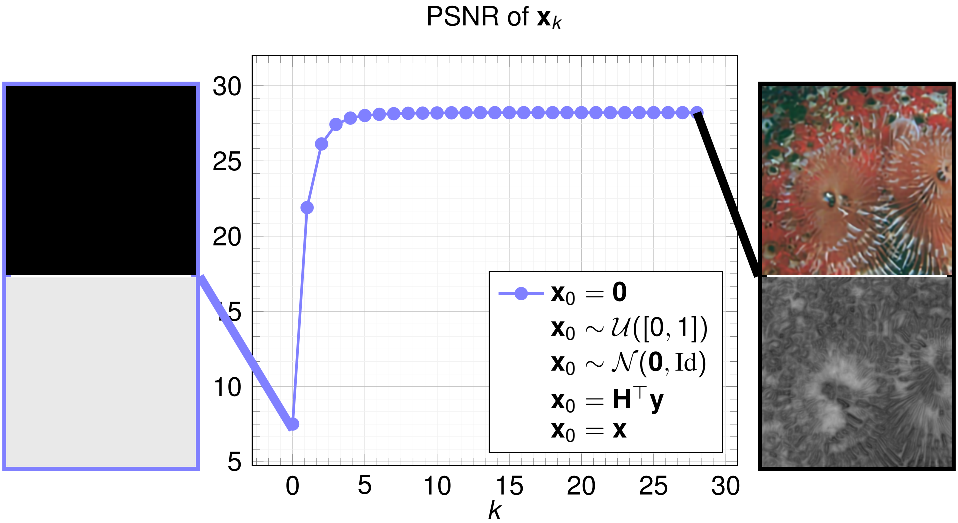
$$\|\mathbf{x}_k - \hat{\mathbf{x}}\|_2 \leq q^{k-1}\|\mathbf{x}_1 - \mathbf{x}_0\|_2.$$

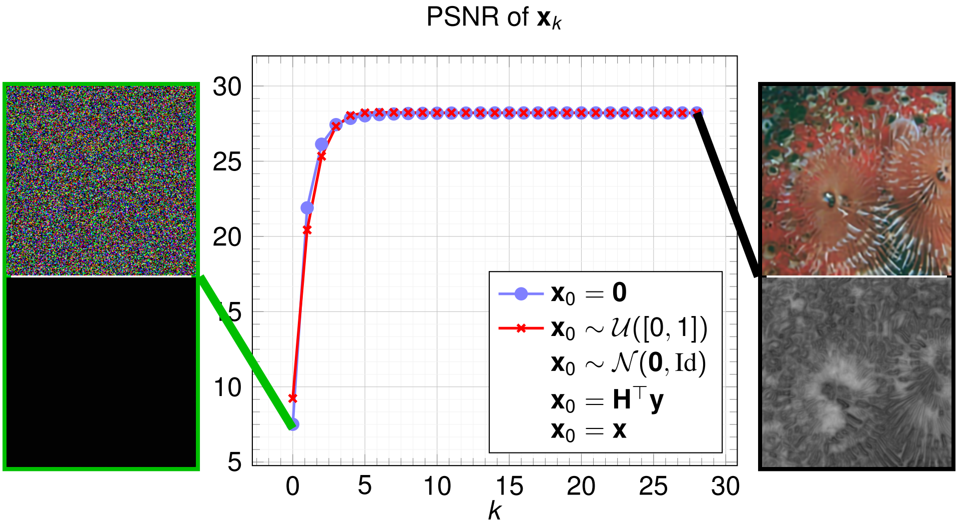
In particular, we have exponential convergence.

Moreover, if $\hat{\mathbf{x}} = \mathcal{T}(\hat{\mathbf{x}}, \mathbf{y}_1)$ and $\hat{\mathbf{z}} = \mathcal{T}(\hat{\mathbf{z}}, \mathbf{y}_2)$, then it holds that

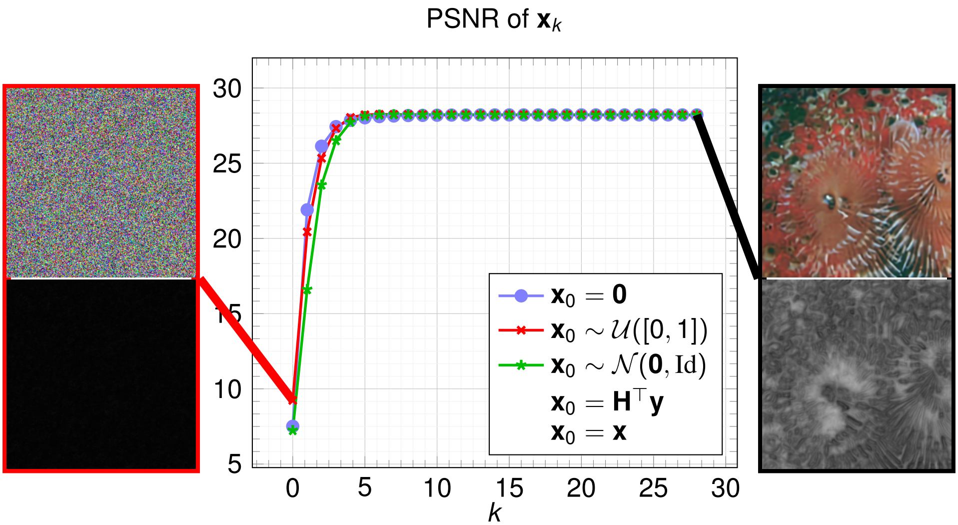
$$\|\hat{\mathbf{x}} - \hat{\mathbf{z}}\| \leq \frac{1}{1-q} \frac{\|\mathbf{H}\|_2}{\lambda_\epsilon} \|\mathbf{y}_1 - \mathbf{y}_2\|_2.$$

DEAL – Stability

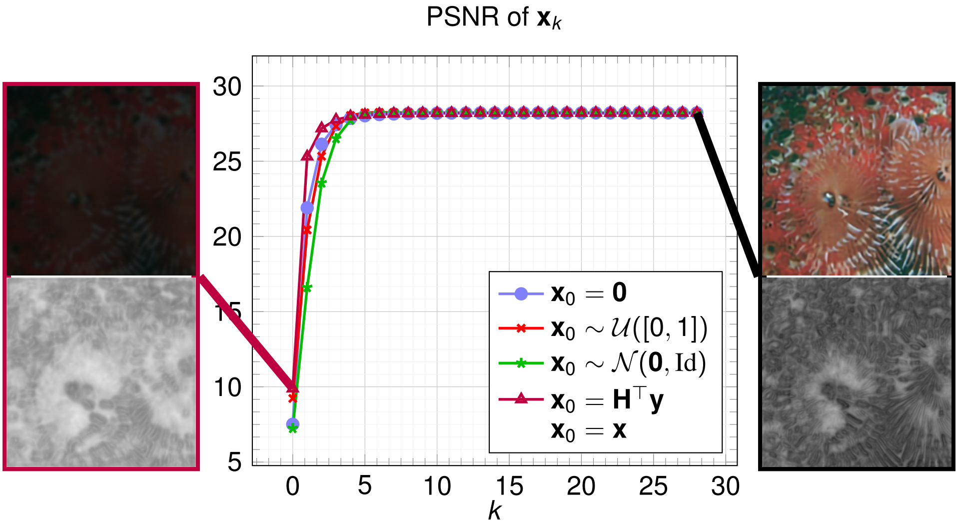



PSNR of $\mathbf{x}_k$
30
25
20
15
10
5
0
5
10
15
20
25
30
k
$\mathbf{x}_0 = \mathbf{0}$
$\mathbf{x}_0 \sim \mathcal{U}([0,1])$
$\mathbf{x}_0 \sim \mathcal{N}(\mathbf{0}, \mathrm{Id})$
$\mathbf{x}_0 = \mathbf{H}^\top \mathbf{y}$
$\mathbf{x}_0 = \mathbf{x}$

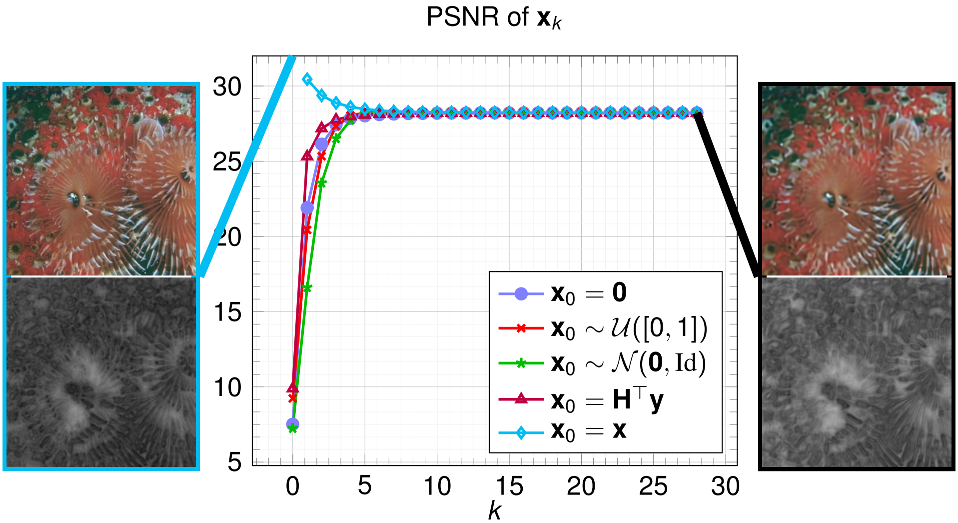
DEAL – Stability



DEAL – Stability

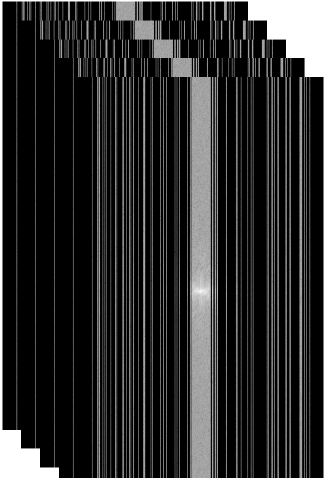


DEAL – Stability



DEAL - Magentic Resonance Imaging

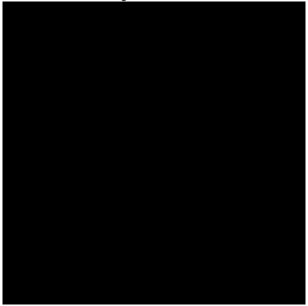
$$\mathcal{T}(\mathbf{x}_k, \mathbf{y}) = \operatorname*{argmin}_{\mathbf{x}} \left(\|\mathbf{Hx} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{Wx}\|^2_{\mathbf{M}(\mathbf{x}_k)} \right)$$



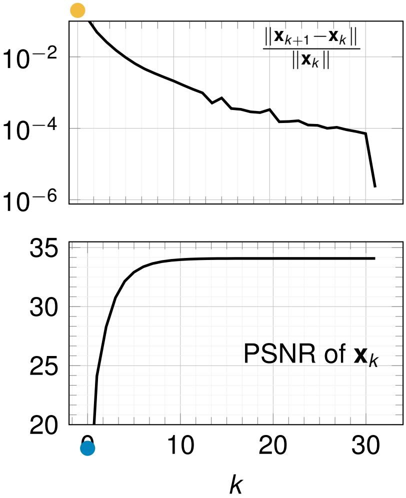
$$H_i : \mathbf{x} \mapsto \mathbf{m} \odot \mathcal{F}(\mathbf{c} \odot \mathbf{x})$$

$$\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathrm{Id})$$

$\mathbf{x}_0$ 10.56

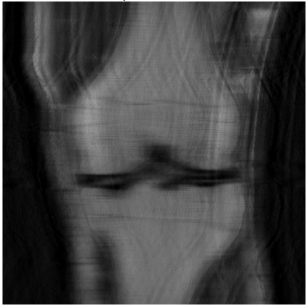


$\overline{\mathbf{M}}(\mathbf{x}_0)$

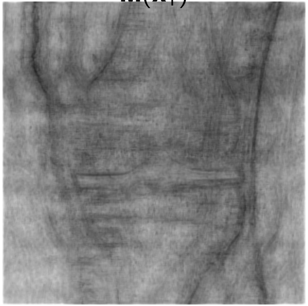


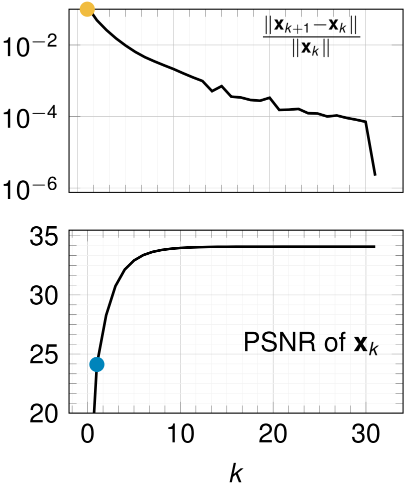
MRI Reconstruction

$\mathbf{x}_1$ 24.12



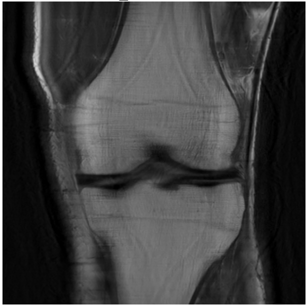
$\overline{\mathbf{M}}(\mathbf{x}_1)$



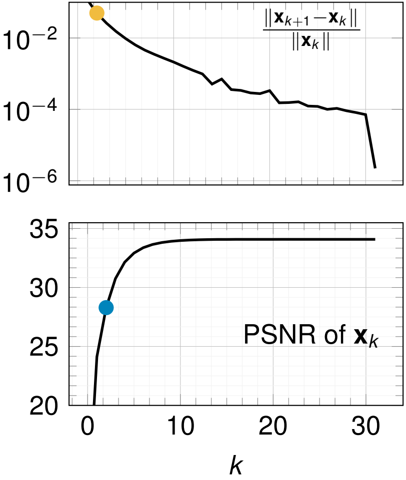


MRI Reconstruction

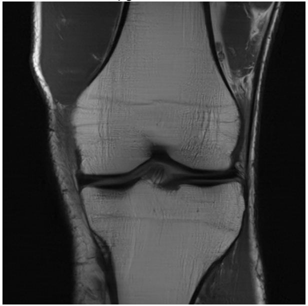
$\mathbf{x}_2$ 28.29



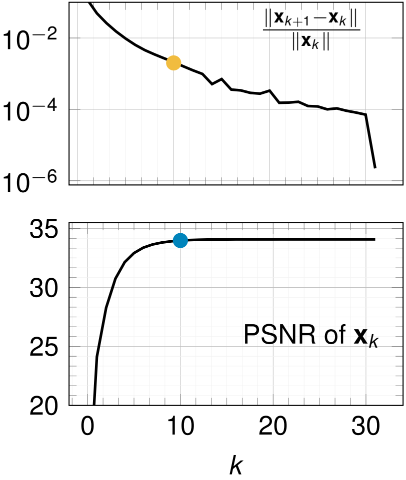
$\overline{\mathbf{M}}(\mathbf{x}_2)$



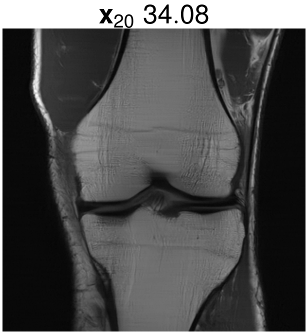
$\mathbf{x}_{10}$ 33.98

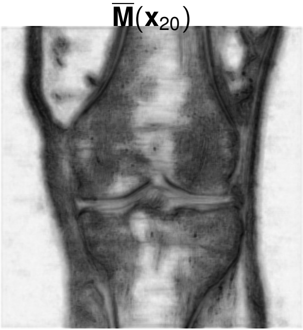


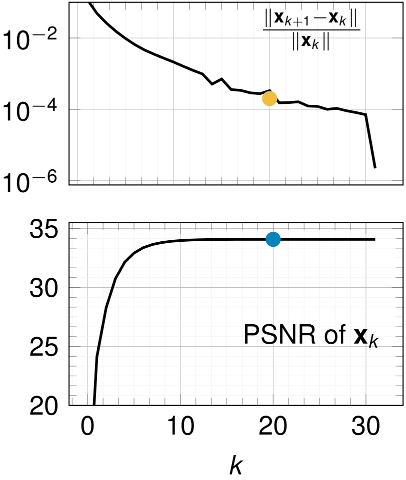
$\overline{\mathbf{M}}(\mathbf{x}_{10})$



MRI Reconstruction

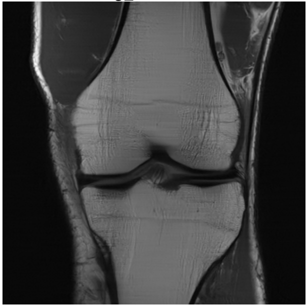




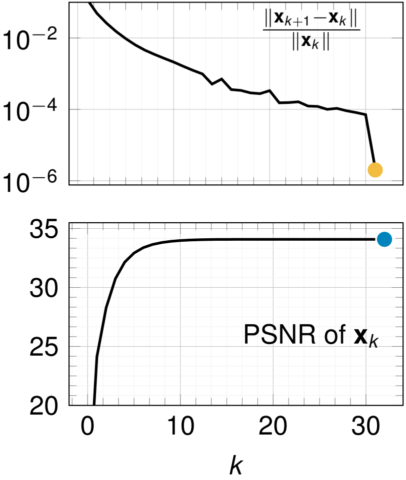


MRI Reconstruction

$\mathbf{x}_{32}$ 34.08



$\overline{\mathbf{M}}(\mathbf{x}_{32})$

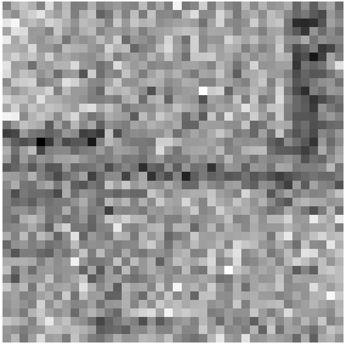


PSNR values for the MRI experiment

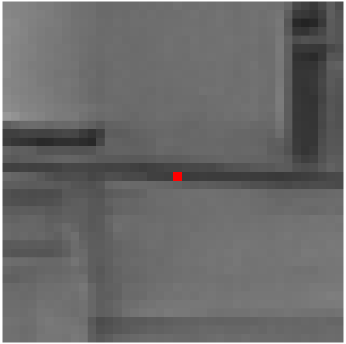
	4-fold single coil		8-fold multi-coil	
	PD	PDFS	PD	PDFS
Zero-fill ($\mathbf{H}^\top \mathbf{y}$)	27.40	29.68	23.80	27.19
TV	32.44	32.67	32.77	33.38
CRR	33.99	33.75	34.29	34.50
WCRR	35.78	34.63	35.57	35.16
SAFI	<u>36.43</u>	<u>34.92</u>	<u>36.06</u>	**35.36**
DEAL (Ours)	**36.59**	<u>34.92</u>	**36.21**	<u>35.32</u>
ProxDRUNet	36.20	**35.05**	35.82	35.12
PnP-DnCNN	35.24	34.63	35.11	35.14

DEAL – Interpretation

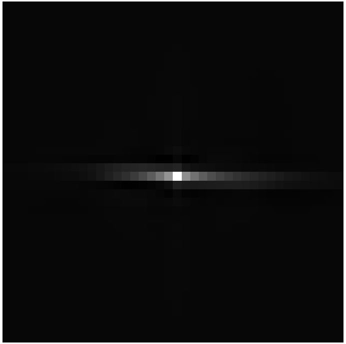
$\mathbf{y}$



$\mathbf{x}_k = \mathbf{A}_{k-1}^{-1}\mathbf{y}$

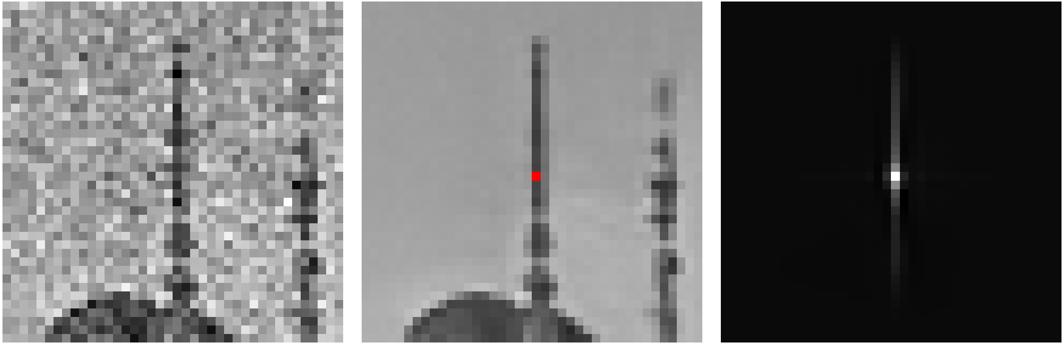


$\mathbf{A}_{k-1}^{-1}\mathbf{e}_n$



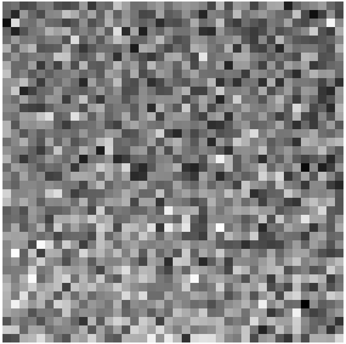
DEAL – Interpretation

$\mathbf{y}$ $\qquad$ $\mathbf{x}_k = \mathbf{A}_{k-1}^{-1}\mathbf{y}$ $\qquad$ $\mathbf{A}_{k-1}^{-1}\mathbf{e}_n$

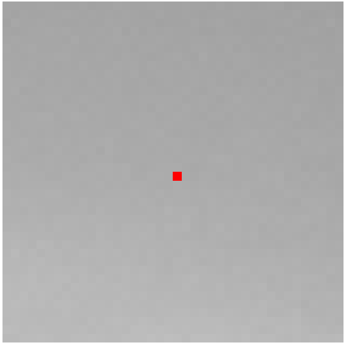


DEAL – Interpretation

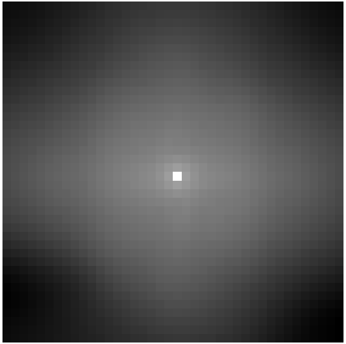
$\mathbf{y}$



$\mathbf{x}_k = \mathbf{A}_{k-1}^{-1}\mathbf{y}$



$\mathbf{A}_{k-1}^{-1}\mathbf{e}_n$



Thank you for your attention!